Γ-convergence: an application to eigenvalue problems

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Abstract. We apply an elementry result of $\Gamma-$convergence to show the dependance on $p(\cdot)$ of the first eigenvalue $\lambda_{p(x)}$ of the $p(x)-$Laplacian problem

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

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1. Introduction and main result

The eigenvalue problem associated with the $p(x)-$Laplacian is defined by

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \ \lambda|u|^{p(x)-2}u, \\ u \in W_0^{1,p(x)}(\Omega), \end{cases} \tag{1.1}$$

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where Ω is a bounded set of $\mathbb{R}^N$ with Lipschitzien boundary, $N \geq 2$ and $p(\cdot) \in \mathcal{P}^{log}(\Omega)$ which is defined by

$$\mathcal{P}^{log}(\Omega) = \{p(\cdot) \in \mathcal{P}(\Omega)/\ p(\cdot)\ is\ \log-\text{Hölder continuous}\}$$

$$\mathcal{P}(\Omega) = \{p(\cdot) \in C(\overline{\Omega})/\ 1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq \sup_{x\in\overline{\Omega}} p(x) =: p^+ < N\ \forall x \in \overline{\Omega}\}.$$

Recall that $p(\cdot)$ is $\log-$Hölder continuous if

$$|p(x) - p(y)| \leq\ -\frac{L}{\log(|x-y|)}$$

for some $L > 0$ and for all $x, y \in \Omega$, with $0 < |x-y| \leq \frac{1}{2}$.

A pair $(u, \lambda) \in W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ is a weak solution of (1.1) provided that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\ dx = \lambda \int_\Omega |u|^{p(x)-2} uv\ dx\ \forall v \in W_0^{1,p(x)}(\Omega) \tag{1.2}$$

u is called an eigenfunction and λ is called the associated eigenvalue.

(1.2) is the Euler-Lagrange equation associated to the minimization of the Rayleigh ratio $R(u) := \frac{\int_\Omega |\nabla u|^{p(x)}\ dx}{\int_\Omega\ |u|^{p(x)}\ dx}$.

In [7], the authors showed that problem (1.1) has an infinite sequence of eigenvalues defined by

$$\lambda^{m,p(x)}(u) = \inf_{K \in \mathcal{W}_{m,p(\cdot)}(\Omega)} \sup_{u\in K} \int_\Omega |\nabla u|^{p(x)}\ dx, \tag{1.3}$$

where $\mathcal{W}_{m,p(\cdot)}(\Omega)$ is the set of symmetric and compact subsets of $\{u \in W_0^{1,p(x)}(\Omega) : \int_\Omega |u|^{p(x)}\ dx = 1\}$ such that $i(K) \geq m$, and i denotes the Krasnoselskii's genus.[1]

When $m = 1$ we use the notation $\lambda^{p(\cdot)} = \lambda^{1,p(\cdot)}$.

The normalization condition $\int_\Omega |u|^{p(x)}\ dx = 1$ taken on u is necessary to avoid the fact that $\lambda^{m,p(\cdot)}(u) \equiv 0$. Indeed, the example introduced in [8] shows that this may happen even if $u \not\equiv 0$.

The first eigenvalue (i.e. $m = 1$) is characterized to be

$$\lambda^{p(\cdot)} = \inf_{u\in\mathcal{M}} \int_\Omega |\nabla u|^{p(x)}\ dx\ ;\ \mathcal{M} = \{u \in W_0^{1,p(x)}(\Omega)/\ \int_\Omega |u|^{p(x)}\ dx = 1\}$$

is a closed manifold.

In the constant exponent case E. Parini showed in [11] the continuity with respect to p of the m$^{\text{th}}$ variational eigenvalue in a sufficiently regular domain.

1. Let E be a real Banach space and $\Sigma(E)$ the set of all closed subsets of E which do not contain 0 and are symmetric with respect to 0. The Krasnoselkii's genus $i(A)$ of a non empty set $A \in \Sigma(A)$ is defined to be the smallest integer $m \in \mathbb{N}$ for which there exists an odd mapping $g \in C(A, \mathbb{R}^m \setminus \{0\})$. If no such mapping exists we write $i(A) = +\infty$.

More precisely the authors require the domain to be bounded with Lipschitz boundary.

A few years later, M. Degiovanni and M. Marzocchi showed in [5] the continuity of the m$^{\text{th}}$ variational eigenvalue but requiring the domain to be only bounded open and connected subset of $\mathbb{R}^N$.

For other articles treating the stability see for example [6],[2],[10] and the references therein.

As far as we know, the stability of eigenvalues associated to the $p(x)-$Laplacian problem have not been treated before. Our main goal in this work is to show that the first eigenvalue associated to the $p(x)-$Laplacian problem $\lambda^{p(\cdot)}$ is continuous with respect to $p(\cdot)$, i.e. for all $h \in \mathbb{N}$

$$\lambda^{p_h(\cdot)} \longrightarrow \lambda^{p(\cdot)} \ as \ p_h(\cdot) \underset{h\to\infty}{\longrightarrow} p(\cdot) \ uniformly.$$

To achieve this goal we utilized an elementary result found in [3] concerning convergence of minimum values of a family of functionals F_h to the minimum value of a limit F.

Since it has been introduced in 1985 by De Giorgi, De Giorgi's $\Gamma-$convergence covers almost all other variational convergences in term of applications as it plays a central role for its compactness properties and for the large number of results concerning $\Gamma-$limits of integral functionals.

The most difficult issue in the study of the Rayleigh ratio with integrals is the non homogeneity of the integrals in marked contrast to the ratio with norms.

Plan of the paper. In Section 2, we collect various preliminaries necessary to apprehend the following section. In section 3 we will prove the main result, that is, the first eigenvalue is continuous with respect to uniform variations of $p(\cdot)$.

2. Preliminaries

2.1 Lebesgue and Sobolev spaces with variable exponents

We refer to the monograph [4] for basics of the spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, but remind their definitions and some important properties.

Let Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitzien boundary and $p(\cdot) \in \mathcal{P}(\Omega)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \ dx$$

is finite. We define the Luxembourg norm on this space by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \ \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is the space of $L^{p(\cdot)}(\Omega)$ functions u whose distributional gradient exists and satisfies $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

Under the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \ \forall u \in W^{1,p(x)}(\Omega)$$

$W^{1,p(\cdot)}(\Omega)$ is a Banach space.

Note that the variable exponent Sobolev spaces resembles classical Sobolev spaces in many aspects: they are Banach spaces and they are reflexive and separable if and only if $1 < p^- \leq p^+ < \infty$.

We require $p(\cdot)$ to be in addition in $\mathcal{P}^{\log}(\Omega)$ to define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ (the space of indefinitely derivable functions with compact support) in $W^{1,p(x)}(\Omega)$ (see [4], Corollary 11.2.4).

Proposition 2.1. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^N)$ be bounded, let $q \in \mathcal{P}(\mathbb{R}^N)$ and suppose that*

$$q(x) \leq p^*(x) - \frac{\omega(|x|)}{log(e + \frac{1}{|x|})}$$

where $\omega: \ [0, \ \infty) \to [0, \ \infty)$ is increasing and continuous with $\omega(0) = 0$.

Then the embedding $W^{1,p(.)}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{q(x)}(\mathbb{R}^N)$ is compact and continuous. with $p^*(x) := \frac{Np(x)}{N-p(x)}$.

1. *(**Poincaré inequality.**) There is a positive constant $C > 0$ such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \ \forall u \in W_0^{1,p(x)}(\Omega).$$

This implies that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

2.2 $\Gamma-$convergence

We recall from [3] basic definitions concerning $\Gamma-$convergence.

Definition 2.2. *Let X be a metric space. A sequence F_h of functionals $(F_h): X \to \overline{\mathbb{R}}$ is said to $\Gamma(X)$ converges to $F : X \to \overline{\mathbb{R}}$ and we write $\Gamma(X) - \lim_{h\to\infty} F_h = F$ if the following hold*

i. *(liminf inequality) For every $u \in X$ and $(u_h) \subset X$ such that $u_h \to u$ in X we have*

$$F(u) \leq \liminf_{h\to\infty} F_h(u_h).$$

ii. *(limsup inequality) For every $u \in X$ there exists a sequence $(u_h) \subset X$ (called recovery sequence) such that $u_h \to u$ in X and*

$$F(u) \geq \limsup_{h\to\infty} F_h(u_h).$$

The condition ii. is characterized by the following (see [9]).

Proposition 2.3. *Let X be a topological space that satisfies the 1^{st} axiom of countability*[2] *and assume that (u_h) is a sequence such that $u_h \to u$ in X as $h \to \infty$, $\limsup_{h\to\infty} F(u_h) \leq F(u)$ and such that for every $m \in \mathbb{N}$ there exists a sequence $\{u_{m,h}\}_h$, $u_{m,h} \to u_m$ as $h \to \infty$ with $\limsup_{h\to\infty} F_h(u_{m,h}) \leq F(u_m)$, then there exists a recovering sequence of u in the sense of ii. of the preceding definition.*

Definition 2.4. *We say that a sequence (F_h) is equicoercive (on X) if for every $t \in \mathbb{R}$ there exists a closed countably compact subset K_t of X such that $\{F_h \leq t\} \subseteq K_t$ for all $h \in \mathbb{N}$.*

Proposition 2.5 ([3], Proposition 7.7)**.** *The sequence (F_h) is equicoercive if and only if there exists a lower semi-continuous coercive function $\psi : X \to \overline{\mathbb{R}}$ such that $F_h \geq \psi$ on X for all $h \in \mathbb{N}$.*

Theorem 2.6 ([3], Theorem 7.8)**.** *Suppose that (F_h) is equicoercive on X, then the $\underbrace{\Gamma - \limsup_{h\to\infty} F_h}_{F'}$ and $\underbrace{\Gamma - \liminf_{h\to\infty} F_h}_{F''}$ are coercive and*

$$\min_{x\in X} F'(x) = \liminf_{h\to\infty} \inf_{x\in X} F_h(x).$$

If in addition (F_h) $\Gamma-$converges to a function F in X then F is coercive and

$$\min_{x\in X} F(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x).$$

3. Main result

$F: \ M \to [0, +\infty]$ defined by:

$$F(u) = \int_\Omega |\nabla u|^{p(x)}\, dx \tag{3.1}$$

and $F_h: \ M_h \to [0, +\infty]$ defined by:

$$\int_\Omega |\nabla u|^{p_h(x)}\, dx. \tag{3.2}$$

To show the continuity of the first eigenvalue we have to prove the following convergence

$$\inf_{u\in M_h} \int_\Omega |\nabla u|^{p_h(x)}\, dx \longrightarrow \inf_{u\in M} \int_\Omega |\nabla u|^{p(x)}\, dx \tag{3.3}$$

as $p_h(\cdot) \to p(\cdot)$ uniformly, where $M_h = \{u \in W_0^{1,p_h(\cdot)}(\Omega) / \ \int_\Omega |u|^{p_h(x)}\, dx = 1\}$ and $M = \{u \in W_0^{1,p(\cdot)}(\Omega) / \ \int_\Omega |u|^{p(x)}\, dx = 1\}$.

2. A topological space satisfies the first axiom of countability if the defining system of neighbourhoods of every point has a countable base.

In view of Theorem 2.6, the convergence (3.3) will be attained if we show that the family F_h $\Gamma-$converges to F as $h \to \infty$ and F_h is equicoercive. Our main concern in this section is to prove the following theorem:

Theorem 3.1. *1. The family of functions (F_h) is equicoercive.*

2. Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded set with Lipschitzien boundary and let $(p_h) \subset \mathcal{P}^{\log}(\Omega)$ and $p \in \mathcal{P}^{\log}(\Omega)$ such that $p_h \to p$ uniformly in Ω. Then

$$\Gamma(L^{p(\cdot)})(\Omega) - \lim_{h\to\infty} F_h = F.$$

Proof. 1. For all $u \in X$ such that $\|u\|_X \geq 1$ we have $\frac{F(u)}{\|u\|_X} \geq \|u\|_X^{p_- -1}$. This implies

$$\Rightarrow \lim_{\|u\|_X\to\infty} \frac{F(u)}{\|u\|_X} = \infty.$$

So F is coercive and clearly lower semi-continuous so we deduce the equicoerciveness by simply taking $\Psi = F$ in Proposition 2.5.

2. `liminf inequality`.

Let $u_h \to u$ in $L^{p(\cdot)}(\Omega)$. If $\liminf\limits_{h\to\infty} F_h(u_h) = +\infty$ there is nothing to prove. Thus, we may assume, without loss of generality, that $u_h \in W^{1,p_h(\cdot)}(\Omega)$ and, up to a subsequence

$$\liminf_{h\to\infty} F_h(u_h) = \lim_{h\to\infty} F_h(u_h) < +\infty.$$

Since $p_h \to p$ uniformly, $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ $\forall h \geq N_\epsilon$ $|p_h - p| < \epsilon$

$$\Rightarrow \; p(x) - \epsilon < p_h(x) \;\forall x \in \Omega \;\forall h \geq N_\epsilon \tag{3.4}$$

Thus, for all $h \geq N_\epsilon$ we have $W_0^{1,p_h(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)-\epsilon}(\Omega)$ and

$$\|\nabla u_h\|_{p(\cdot)-\epsilon} \leq (1 + |\Omega|)\|\nabla u_h\|_{p_h(\cdot)} \leq D, \tag{3.5}$$

where D is a constant not depending on ϵ.

Then(u_h) is a bounded sequence in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$ which is reflexive, hence there exists a subsequence (u_h) such that $u_h \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$ with $u \in W_0^{1,p(\cdot)-\epsilon}(\Omega)$ for all $\epsilon > 0$ sufficiently small.

We shall next show that the limit $u \in W_0^{1,p(\cdot)}(\Omega)$ for ϵ sufficiently small.

By the lower semicontinuity of the modular and inequality (3.5) we have

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(x)-\epsilon}\, dx &\leq \liminf_{h\to\infty} \int_\Omega |\nabla u_h|^{p(x)-\epsilon}\, dx \\
&\leq \liminf_{h\to\infty} \max\{\|\nabla u_h\|_{p(\cdot)-\epsilon}^{p^+-\epsilon}, \|\nabla u_h\|_{p(\cdot)-\epsilon}^{p^--\epsilon}\} \\
&\leq \liminf_{h\to\infty} \max\left\{\left[(1+|\Omega|)|\nabla u_h|_{p_h(\cdot)}\right]^{p^+-\epsilon},\right. \\
&\left.\left[(1+|\Omega|)|\nabla u_h|_{p_h(\cdot)}\right]^{p^--\epsilon}\right\} \\
&\leq \max\left\{D^{p^+-\epsilon}, D^{p^--\epsilon}\right\} < +\infty.
\end{aligned}$$

Thus

$$\sup_{\epsilon>0}\int_\Omega |\nabla u|^{p(x)-\epsilon}\,dx < +\infty$$

and by Fatou's lemma

$$|\nabla u|^{p(x)} = \liminf_{\epsilon\to 0}|\nabla u|^{p(x)-\epsilon} \in L^1(\Omega).$$

It follows that

$$u \in W_0^{1,p(\cdot)}(\Omega).$$

By Young's inequality and (3.4) we have

$$\int_\Omega |\nabla u_h|^{p(x)-\epsilon}\,dx \le \int_\Omega \frac{p(x)-\epsilon}{p_h(x)}|\nabla u_h|^{p_h(x)}\,dx + \int_\Omega \frac{p_h(x)-p(x)+\epsilon}{p_h(x)}\,dx$$
$$\le \int_\Omega |\nabla u_h|^{p_h(x)}\,dx + \epsilon\int_\Omega \frac{1}{p_h(x)}\,dx + \frac{1}{p_h^-}\|p_h - p\|_\infty.$$

In view of the weak convergence to u in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$ and the uniform convergence of p_h to p we have

$$\int_\Omega |\nabla u|^{p(x)-\epsilon}\,dx \le \liminf_{h\to\infty}\int_\Omega |\nabla u_h|^{p_h(x)}\,dx + \epsilon\frac{|\Omega|}{p^-}.$$

Since the inequality holds for each $\epsilon > 0$, letting $\epsilon \searrow 0$ and applying Fatou's lemma we obtain

$$\int_\Omega |\nabla u|^{p(x)}\,dx \le \liminf_{h\to\infty}\int_\Omega |\nabla u_h|^{p_h(x)}\,dx.$$

`limsup inequality`

Suppose that $u \in W_0^{1,p(\cdot)}(\Omega)$ (if $u \notin W_0^{1,p(\cdot)}(\Omega)$ $F_h(u) = +\infty$ and there is nothing to prove). Since $p(\cdot)$ is log −Hölder continuous $\overline{C_0^\infty(\Omega)} = W_0^{1,p(\cdot)}(\Omega)$ and then it exists a sequence $(u_n) \subset C_0^\infty(\Omega)$ such that

$$u_n \to u \in W_0^{1,p(\cdot)}(\Omega)$$

$$\text{i.e } \int_\Omega |\nabla u|^{p(x)}\,dx = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p(x)}\,dx. \tag{3.6}$$

On the other hand, for each $m \in \mathbb{N}$

$$|u_m|^{p_h(x)} \longrightarrow |u_m|^{p(x)} \text{ as } h\to\infty \text{ for a.e } x\in\Omega$$

and by the uniform convergence of p_h to p $\forall\epsilon > 0$ we can find $N_\epsilon \in \mathbb{N}$ such that $p_h < p + \epsilon$ $\forall$ $h \ge N_\epsilon$ and then

$$|u_m|^{p_h(x)} \le |u_m|^{p(x)+\epsilon} + 1 \in L^1(\Omega)\ \forall\ x\in\Omega\ \forall\ h \ge N_\epsilon,$$

whence by the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_\Omega |u_m|^{p_h(x)}\,dx = \int_\Omega |u_m|^{p(x)}\,dx\ \forall m\in\mathbb{N}.$$

The proof is completed by Proposition 2.3 . □

Theorem 3.2. *The first eigenvalue associated to the $p(x)-$Laplacian $\lambda^{p(\cdot)}$ is continuous with respect to the exponent $p(\cdot)$.*

Proof. By Theorem 3.1 the functional F defined in (3.2) is the $\Gamma-$limit of F_h and since the functionals F_h are coercive $\forall\ h \in \mathbb{N}$ by Theorem 2.6

$$\inf_{u \in M_h} F_h \longrightarrow \inf_{u \in M} F \text{ as } h \to \infty.$$

Define $\lambda^{p_h(\cdot)} := \inf\limits_{u \in M_h} F_h(u)$ and $\lambda^{p(\cdot)} := \inf\limits_{u \in M} F(u)$ we conclude that

$$\lambda^{p_h(\cdot)} \longrightarrow \lambda^{p(\cdot)}$$

when $p_h(\cdot) \to p(\cdot)$ uniformly. □

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